Problems with the definition of renormalized Hamiltonians for momentum-space renormalization transformations

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For classical lattice systems with finite (Ising) spins, we show that the implementation of momentum-space renormalization at the level of Hamiltonians runs into the same type of difficulties as found for real-space transformations: Renormalized Hamiltonians are ill-defined in certain regions of the phase diagram. [S1063-651X(99)14505-7]

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I. INTRODUCTION

Despite the great success of renormalization-group (RG) ideas, both for computations and as a heuristic guide, many aspects of the theory still lack rigorous mathematical justification. The filling of this gap is more than just of academic interest. It has been repeatedly pointed out (e.g. [[1], p. 82], [[2], footnote on p. 38], [[3], p. 268]) that the method is not a black-box type of technique; its successful application requires some understanding of the underlying physics or one may be led to incorrect conclusions. Studies on the foundations of real-space transformations [4–8] suggest that a similar remark applies to the underlying mathematics. Indeed, these studies show that in various occasions renormalized Hamiltonians are ill-defined. The finite-volume probabilities of the renormalized system exhibit a long-range dependence on boundary spins that is incompatible with the existence of a Hamiltonian, at least one defined in the usual (summable) sense. Such a ''pathology'' is usually referred to as *non-Gibbsianness*. This phenomenon, which appears after a *single* application of the transformation, was first detected in the vicinity of first-order phase transitions, but was later discoverd in other regions of phase diagrams, including at high magnetic fields [8,9] and at high temperatures [9,10]. It follows that the design of the renormalization transformation is crucial for the very *existence* of a renormalization flow in a suitable space.

Nevertheless, the lack of similar studies for momentum-space transformations left open the possibility that they could be free of this pathological behavior. That is, the question remained as to whether such transformations, possibly with a soft cutoff, would generally lead to an actual renormalized Hamiltonian [5,11]. In this paper we present a simple example showing that this is in general *not* the case, as already suspected by Griffiths [12]. There is no essential difference between real-space and momentum-space transformations, at least in the case in which the spins are bounded. The same mechanism—the existence of a phase transition in the system of original spins, constrained by a particular block-spin configuration— causes similar problems with the definition of renormalized Hamiltonians. For an earlier suggestion that momentum-space maps are not all that different from real-space maps, see also [13].

As we remark at the end of Sec. IV, these problems can be interpreted as a manifestation of the well-known ''large-field problem.'' It might be hoped that the considerable experience accumulated in the treatment of this type of problem could be of help to control the non-Gibbsianness ''pathologies.''

II. MOMENTUM TRANSFORMATIONS

We consider Ising spins $\sigma_{\vec{x}}=-1,+1$ on a lattice, $\vec{x}\in\mathbb{Z}^d$. For each finite periodic cube $V=[-N,N]^d$ in $\mathbb{Z}^d$ we define the Fourier-transformed variables

$$\hat{\sigma}^V_{\vec{k}} := \sum_{\vec{x}\in V} \sigma_{\vec{x}} e^{-i\vec{k}\cdot\vec{x}}, \tag{2.1}$$

where $\vec{k}\cdot\vec{x} := k_1x_1+\cdots+k_dx_d$, and each k_i belongs to the Brillouin zone: $\mathcal{B}_N=\{-\pi,-\pi[1-1/(2N+1)],\ldots,\pi[1-1/(2N+1)],\pi\}$. The inverse of Eq. (2.1) is

$$\sigma_{\vec{x}}=\frac{1}{(2N+1)^d}\sum_{\vec{k}\in\mathcal{B}_N^d}\hat{\sigma}^V_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} \tag{2.2}$$

for $\vec{x}\in V$.

A *momentum-space transformation* is defined in two steps.

(i) A cutoff is applied to the variables $\hat{\sigma}^V_{\vec{k}}$:

$$\hat{\sigma}'^V_{\vec{k}} := \hat{f}(\vec{k})\hat{\sigma}^V_{\vec{k}}. \tag{2.3}$$

The *volume-independent* cutoff function $\hat{f}$ is designed so as to keep only momenta smaller than a certain threshold k_0.

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(ii) Momenta are rescaled by the factor k_0 so as to return to a Brillouin zone in $[-\pi,\pi]^d$:

$$\hat{\sigma}'^{V}_{\vec{k}'} := \hat{f}(\vec{k}' k_0/\pi)\hat{\sigma}^{V}_{\vec{k}' k_0/\pi}. \tag{2.4}$$

In addition, the renormalized variables $\hat{\sigma}'^{V}_{\vec{k}'}$ are usually rescaled in applications. We will not do this, as we shall not apply the transformation more than once.

In Wilson's original approach [and references therein] [14], the cutoff function $\hat{f}(\vec{k})$ was chosen simply as the step function

$$\chi_{k_0}(\vec{k}) := \begin{cases} 1 & \text{if } |k_i| \leqslant k_0, \quad i=1,\ldots,d \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

It was quickly realized, however, that such a sharp cutoff leads to unwanted long-range terms in the renormalized Hamiltonian (see, e.g., [14], p. 153], [15], Appendix 2]). To avoid such terms one usually takes smooth momentum cutoffs, that is, functions $\hat{f}$ which go to zero in a sufficiently differentiable fashion. Such functions are obtained, for instance, via a convolution

$$\hat{f}(\vec{k}) = \sum_{\vec{\ell} \in \mathcal{B}^d_N} \delta_\Delta(\vec{k}-\vec{\ell})\chi_{k_0}(\vec{k}) \tag{2.6}$$

with a smooth δ-like function δ_Δ peaked at $k=0$ of width Δ.

Rigorously speaking, one is interested in the limit $V \to \mathbb{Z}^d$ of this procedure. To make sense of this limit we return to real space, where the prescription (2.4) translates into the relation

$$\sigma'^{V}_{\vec{x}'} = \sum_{\vec{y} \in V} f^V(L\vec{x}'-\vec{y})\sigma_{\vec{y}}, \quad \vec{x}' \in V/L, \tag{2.7}$$

where f^V is the V-dependent (inverse) discrete Fourier transform of $\hat{f}$ and

$$L := \frac{\pi}{k_0}. \tag{2.8}$$

The volume V is assumed to be a disjoint union of cubes of side L (i.e., N is a multiple of L). As $V \to \mathbb{Z}^d$, the function $f^V(\vec{x})$ tends to

$$f(\vec{x}) := \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi} \hat{f}(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} d\vec{k}. \tag{2.9}$$

A sharp cutoff in momentum space gives rise to a *nonsummable* function f, i.e., $\Sigma_{\vec{x}\in\mathbb{Z}^d}|f(\vec{x})| = \infty$. [The inverse transform of χ_{k_0} is proportional to the function $\Pi_{i=1}^d \sin(k_0 x_i)/(k_0 x_i)$]. Summability is restored if $\hat{f}$ is smooth enough (for example, once differentiable). In such a case, expression (2.7) remains valid in the thermodynamic limit:

$$\sigma'_{\vec{x}'} = \sum_{\vec{y} \in \mathbb{Z}^d} f(L\vec{x}'-\vec{y})\sigma_{\vec{y}}, \tag{2.10}$$

and the renormalized spins remain *bounded* in this limit. They may take a large number of values, but within some finite interval.

Expression (2.10) shows that a cutoff in momentum space, even a smooth one like Eq. (2.6), leads to *nonlocal* averages in real space, i.e., to functions f extending to infinity. This is the distinctive feature with respect to the real-space transformations analyzed, for instance, in [8]. Nevertheless, it is expected that ''the physics behind integration over fluctuations having wave numbers $[|k_i| > k_0]$ is the same as the physics behind the formation of blocks of spins having volume $[L^d]$ in real space'' [[16], Section 4.2]. To ensure this, the momentum-space cutoff should lead to an *almost local* average in real space. That is, the function f should decay rapidly outside of a region of size not much larger than L. We see that if f has a Fourier transform of the type (2.6), the contribution of spins outside the volume of size L is of order $\ln(\Delta/k_0)$. We conclude that the cutoff function $\hat{f}$ must approach zero in a ''gradual'' manner, that is, with Δ of the order of k_0 in Eq. (2.6).

A *soft momentum cutoff* is a function $\hat{f}$ that is smooth and gradual in the above sense.

III. THE PHENOMENON OF NON-GIBBSIANNESS

A state (probability measure or distribution) is called Gibbsian if it can be written in terms of Boltzmann-Gibbs weights for an ''acceptable Hamiltonian [which] ... must satisfy the additional requirement of locality ... [that is,] a quantity that is additive over distant lattice sites'' [[14], p. 145]. In other words, Hamiltonians must be such that the energy of disjoint volumes is additive except for boundary terms whose contribution is small in comparison with the volumes. For classical lattice systems, the appropriate requirement is that the flipping of one spin lead to a *finite* energy change whatever the configuration of the remaining spins is. If the system involves spins forced to satisfy certain conditions (local, like hard-core, or global, as in the example below), the finite-energy-change requirement must be adjusted appropriately because the overall constraint may prevent the flipping of a single spin or of isolated groups of spins. Let us be precise about this.

The Boltzmann-Gibbs weights are constructed via finite-volume Hamiltonians which are, in general, sums of many-body terms: For each finite volume Λ in $\mathbb{Z}^d$ (for instance, a cube), they take the form

$$H_\Lambda(\sigma) = \sum_{B: B\cap\Lambda\neq\varnothing} \Phi_B(\sigma_B), \tag{3.1}$$

where each Φ_B is a (Λ-independent) function only of the spins in the finite set $B \subset \mathbb{Z}^d$, i.e., of the variables $\sigma_B = \{\sigma_{\vec{x}}\}_{\vec{x}\in B}$. For Ising spins, these functions Φ_B are usually written in the form $J_B \Pi_{\vec{x}\in B}\sigma_{\vec{x}}$; the general expression (3.1) is more suitable for spins larger than 1/2, where one would need powers of $\sigma_{\vec{x}}$, and also for some particular spin-1/2 interactions [17]. Obviously certain summability requirements are needed to make sense of formula (3.1), or, equivalently, to ensure that the boundary terms —that is, the terms

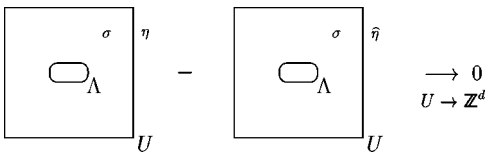

FIG. 1. Test for quasilocality. As U tends to $\mathbb{Z}^d$ while keeping fixed the configuration inside U, the energy inside Λ should asymptotically become independent of the configuration η or $\hat{\eta}$ outside U.

corresponding to sets B that intersect both Λ and its complement— have a small contribution compared with the volume of Λ.

Let us first consider systems of spins not subjected to any local or global conditioning (we are reserving the word ''constraint'' for the ''constrained spin system'' to be introduced below). When the interaction is of finite range there is nothing to impose: For each finite Λ, there are only a finite number of contributing boundary terms, all of which have diameter smaller than the range. For more general systems, involving terms with arbitrarily long range, the default requirement is

$$\sup_{\vec{x}} \sum_{B \ni \vec{x}} \|\Phi_B\| < \infty, \tag{3.2}$$

where $\|\Phi_B\| := \sup_\sigma |\Phi_B(\sigma_B)|$. From the physical point of view, this is the condition, mentioned above, that a single flip produce a finite energy change. From the mathematical point of view, besides ensuring the summability of Eq. (3.1) for every σ, this condition leads to a natural property amounting to ''independence from infinity'' of the Hamiltonians. Indeed, while for non-finite-range systems the dependence of the Hamiltonian (3.1) on boundary spins—that is, on σ_x with $x \notin \Lambda$— may extend to the whole of the complement of Λ, condition (3.2) implies that this dependence must decay with the distance to the region Λ. This property, which is called *quasilocality*, was central to the arguments presented in [6,7]. Let us state it precisely. We take a sequence of cubes $U \supset \Lambda$ with larger and larger radius and *fix* the configuration σ inside U, in particular in the intermediate, ''buffer,'' region $U \setminus \Lambda$ (see Fig. 1). It is not hard to see that the summability condition implies (in fact, it is equivalent to) the following fact. Let us denote $\sigma_U \eta$ the configuration

$$(\sigma_U \eta)_{\vec{x}} = \begin{cases} \sigma_{\vec{x}} & \text{if } \vec{x} \in U \\ \eta_{\vec{x}} & \text{if } \vec{x} \in \mathbb{Z}^d \setminus U. \end{cases} \tag{3.3}$$

Then,

$$\sup_{\eta \hat{\eta}} |H_\Lambda(\sigma_U \eta) - H_\Lambda(\sigma_U \hat{\eta})| \underset{U \to \mathbb{Z}^d}{\to} 0 \tag{3.4}$$

for all configurations σ and all finite regions Λ. That is, asymptotically the Hamiltonian becomes independent of what happens outside U.

The Boltzmann-Gibbs weights

$$\rho_\Lambda(\sigma) := \frac{\exp[-\beta H_\Lambda(\sigma)]}{Z_\Lambda(\sigma)} \tag{3.5}$$

[$Z_\Lambda(\sigma)$ is the obvious normalization factor], constructed with the Hamiltonians (3.1), inherit the quasilocality property (3.4):

$$\sup_{\eta \hat{\eta}} |\rho_\Lambda(\sigma_U \eta) - \rho_\Lambda(\sigma_U \hat{\eta})| \underset{U \to \mathbb{Z}^d}{\to} 0 \tag{3.6}$$

for all cubes Λ and configurations σ. We remark that this ''independence from infinity'' of the finite-volume probability distributions is there regardless of whether the (infinite-volume) system exhibits long-range order or not. For instance, for the Ising model, the left-hand side of Eq. (3.6) is *exactly* zero for all temperatures once U contains the neighbors of Λ, while the long-range-order properties depend on the temperature. From a probabilistic point of view, Eq. (3.6) is saying that the *conditional* finite-volume probabilities are ''insensitive'' to what happens at infinity. The fully infinite-volume distribution need not be so.

In [6,8–10] it is shown that for a number of real-space renormalization transformations in different regions of the phase diagram of Ising and Potts models, this quasilocality property is *violated* for the renormalized weights for some Λ' (usually very small, formed by one or two sites), and some configuration σ'. By the above chain of implications, this shows that in these instances the renormalized weights cannot be written as Boltzmann-Gibbs weights for an *acceptable*—in Wilson's and Kogut's sense—renormalized Hamiltonian. That is, while the renormalized Boltzmann-Gibbs weights

$$\rho'_{\Lambda'}(\sigma') = \frac{1}{Z_{\Lambda'}(\sigma')} \text{``} \sum_{\sigma \to \sigma'} \text{''} \exp[-\beta H_\Lambda(\sigma)] \tag{3.7}$$

are always well defined, it is *not* true that the family of identities

$$\exp[-\beta' H'_{\Lambda'}(\sigma')] := \text{``} \sum_{\sigma \to \sigma'} \text{''} \exp[-\beta H_\Lambda(\sigma)] \tag{3.8}$$

give rise to Hamiltonians H' that can be written in the form (3.1) for a suitable interaction that *satisfies the summability condition* (3.2). This is the phenomenon of non-Gibbsianness referred to in the title of this section.

The symbol ''Σ'' in Eqs. (3.7) and (3.8) is a reminder that the operation involved may not be a standard sum because there may be uncountably many original configurations σ leading to the same renormalized configuration σ'. In these cases, the operation is rather a sum combined with a suitable limit procedure, or, in mathematical terms, an integral with respect to the product measure $\Pi_{\vec{x} \in \mathbb{Z}^d}[(1/2)\Sigma_{\eta_{\vec{x}}}]$. [The reader interested in the rigorous construction of Eq. (3.7) is referred to the discussion in [[8], pp. 987–990]. On the other hand, the notation $\sigma \to \sigma'$ represents the space of *original* spins *constrained* to produce the indicated renormalized spin σ'. In this paper we reserve the name *constrained system* for such a system of original spins.

The preceding discussion has to be slightly adapted for the case of the momentum transformations (2.10), because they lead to spin configurations subjected to the global condition of being images of Eq. (2.10). For the sake of brevity,

let us call *profiles* these image —or renormalized— configurations. They have spin values

$$-\sum_{\vec{y}\in\mathbb{Z}^d}|f(\vec{y})|\leqslant\sigma'_{\vec{x}'}\leqslant\sum_{\vec{y}\in\mathbb{Z}^d}|f(\vec{y})|, \tag{3.9}$$

and they form a space with a rather cumbersome structure. For instance, it is in general impossible to find different profiles taking *exactly* the same values on a (finite or infinite) set U. The notion of quasilocality, Eq. (3.4) or (3.6), loses, therefore, its original meaning. Rather, the ''independence from infinity'' should be understood as the following *continuity* property: Given any $\varepsilon>0$ one has that for $U'\subset\mathbb{Z}^d$ large enough and $\delta>0$ small enough

$$|\sigma'_{\vec{x}'}-\omega'_{\vec{x}'}|<\delta,\vec{x}'\in U'\Rightarrow|\rho'_{\Lambda'}(\sigma')-\rho'_{\Lambda'}(\omega')|\leqslant\varepsilon. \tag{3.10}$$

This property is satisfied, for instance, if the renormalized Hamiltonian on the profiles is of the form (3.1) for an interaction that is summable in the sense (3.2). The purpose of this paper is to show that there exists one momentum transformation such that, at least at low temperatures, the renormalized system lacks property (3.10), and hence there is no renormalized Hamiltonian defined in the usual sense. We shall determine, in Sec. IV, *one* particular profile σ' (equal to ''all-0'') for which Eq. (3.10) is violated for a particular Λ' (formed by two consecutive sites). Of course, it is natural to wonder how relevant the phenomenon is from the physical point of view, given that the discontinuity involves few and atypical configurations σ'. We shall comment on this point in Sec. V.

The lack of continuity (3.10) [or quasilocality (3.6)] can be interpreted as exhibiting some sort of ''action at a distance'': Infinitely far away spin-flips produce a sizeable change close to the origin, *even when the intermediate renormalized spins are (almost) frozen*. This is in contrast with the usual behavior in equilibrium statistical mechanics (Gibbsian behavior) where changes at infinity can propagate only through fluctuations of intermediate spins. It is not hard to imagine the explanation: A fixed renormalized configuration still allows fluctuations in the corresponding constrained system of original spins. These fluctuations act as ''hidden degrees of freedom'' that in some instances can bring information from infinity. This happens when the constrained system of original spins develops long-range correlations, i.e., when it undergoes a *phase transition*. The argument of the next section consists precisely in showing that for the chosen example such a phase transition does take place.

IV. NON-GIBBSIANNESS DUE TO MOMENTUM TRANSFORMATIONS

We consider the nearest-neighbor ferromagnetic Ising model in $\mathcal{L}=\mathbb{Z}^2$,

$$H=-\sum_{\langle\vec{x},\vec{y}\rangle}\sigma_{\vec{x}}\sigma_{\vec{y}}+h\sum_{\vec{y}}\sigma_{\vec{y}}, \tag{4.1}$$

at low temperatures, that is, large β. It has been shown that the low-temperature states for this model under a (local) block-average transformation with even block sizes are mapped onto non-Gibbsian states [[8], Theorem 4.6]. A very simple example of this phenomenon for 1 by 2 blocks was presented in [18]. We shall now prove a similar result for a momentum transformation of the type introduced above.

Let us first sketch some intuition behind our argument. In real space, the momentum transformation (2.10) looks approximately like an average. Expressions of this type have been studied, for example, in [19,20] (see also [[15] Appendix 2] for a stochastic version). Even when Eq. (2.10) involves a sum over all spins $\vec{y}$ of the lattice, one would expect that each $\sigma'_{\vec{x}'}$ is essentially determined only by the spins $\sigma_{\vec{y}}$ inside the block of side L centered at $L\vec{x}'$. Therefore, the mechanism causing non-Gibbsianness for average transformations [[8] Section 4.3.5] should apply to the present case with minor adaptations.

We will take for our example the identity in one direction and in the other direction the soft cutoff function:

$$\hat{f}(k)=\begin{cases}\cos^2(k) & \text{for } |k|\leqslant\pi/2,\\ 0 & \text{otherwise.}\end{cases} \tag{4.2}$$

This function integrates out half of the momenta degrees of freedom in this direction, which corresponds to taking a (not strictly local) average over blocks of size 1 by 2, centered at sites with even coordinates in the direction in which we renormalize.

Its Fourier transform is easily computable. Indeed, $f(0)=\frac{1}{4}$, $f(2)=f(-2)=\frac{1}{8}$, and for all other n

$$f(n)=-\frac{2}{\pi}\sin\left(n\frac{\pi}{2}\right)\times\frac{1}{(n-2)n(n+2)}. \tag{4.3}$$

[In particular $f(n)=0$ for all even $n\neq0,\pm2$.]

The initial (and crucial) part of the argument consists in exhibiting a transformed configuration ω' such that the corresponding constrained system of original spins has a phase transition *at zero temperature*. The configuration in question is $\omega'_{\vec{x}}=0$ for all $\vec{x}\in\mathbb{Z}^2$. The corresponding original configurations must, therefore, satisfy the constraint

$$\sum_l f(l)\omega_{2n+l}=0 \tag{4.4}$$

for each n in the direction under consideration. We claim that the only four ground states are the 4-periodic configuration (strip state)

$$++--++--++ \tag{4.5}$$

and its translates over distances 1, 2, or 3 (while in the other direction they are of course translation invariant). It is immediate to check that these configurations are compatible with the constraint. Moreover, it is not difficult to check that under the constraint (4.4) they are ground states.

Indeed, to lower the energy of the configurations one needs a larger number of consecutive aligned spins. But we claim that if there were a row of three identical spins, say plus, next to each other, then constraint (4.4) could not be satisfied. The idea is that the contribution of these three spins cannot possibly be compensated by the remaining spins so as

to make the sum (4.4) equal to zero. If the middle plus would be on an even site, say at zero, this happens because

$$\sum_{|l|\geqslant 3} |f(l)| < \min\{f(0)+f(1)+f(-1) + \omega_2 f(2)+\omega_{-2}f(-2): \omega_2, \omega_{-2}=\pm 1\} = f(1)+f(-1) \quad \left(=\frac{4}{3\times\pi}\right), \tag{4.6}$$

as a simple calculation shows. If, on the other hand, the middle plus would be on an odd site, the interval of three plus spins would need to have a minus spin both to its left and to its right, otherwise we would have the situation just shown to be impossible. In this case, we see that the constraint (4.4), centered at either the left or the rightmost plus site, cannot be fulfilled because, once more, the ''tail'' cannot compensate the central five spins. For instance, assuming the origin is the rightmost plus of the block, this follows from the fact that

$$\sum_{|l|\geqslant 3} |f(l)| < \min\{f(0)-f(1)+f(-1)+\omega_2 f(2) + f(-2): \omega_{-2}=\pm 1\} = f(0) \quad (=\tfrac{1}{4}). \tag{4.7}$$

We conclude that the constrained system has multiple, namely four, ground states.

The remaining part of the argument follows closely the presentation in [[8] Section 4.3.5]. There are three additional steps.

(1) *Existence of a phase transition at nonzero temperature for the constrained system.* This follows from a well-known theory (Pirogov-Sinai theory [[21], Chap. 2], [[8], Appendix B]; note that as remarked in [22], the theory also applies to systems with constraints). There is one extremal phase associated to each of the four ground states. (Depending on *f*, these constrained phases may involve only a small number of configurations).

(2) *Selection of the phases of the constrained system via block-spin boundary conditions.* This requires the choice of a profile σ'^{+} such that if it is imposed in a sufficiently large (but finite) volume, the constrained configurations deep inside this volume have to be close to the prescribed ground state. This is straightforward, though a little cumbersome to write. The procedure is as follows: For a given (large) region U', pick first a configuration σ^{+} such that (a) inside U' coincides with the strip configuration (4.5) corresponding to the phase to be selected, and (b) outside U', is identical equal to $+1$. The corresponding profile is our σ'^{+}. A calculation involving simple inequalities very much like Eqs. (4.6) and (4.7) shows that σ^{+} is the *only* original configuration yielding the profile σ'^{+}. It was to ensure this uniqueness that the ''all-+'' configuration was chosen for the exterior of U' (less extreme configurations would have destroyed this uniqueness; of course the ''all-−'' configuration would have worked equally well). It is now easy to see that this profile σ'^{+} does the job it was designed for. Indeed, for each set $M' \supset U'$ let us consider the family $O'^{+}_{U',M'}$ formed by all the profiles obtained by performing the momentum transformation (2.10) of all configurations coinciding with σ^{+} inside M'. These profiles have two important properties.

(i) As M' grows, the profiles become arbitrarily close to σ'^{+} inside U'.

(ii) Given any fixed Λ', if $U' \supset \Lambda'$ is large enough, then for $M' \supset U'$ large enough all the constrained configurations for profiles of $O'^{+}_{U',M'}$ coincide with the initially selected striped configuration inside Λ'. This is checked through the same inequalities proving the uniqueness of the original configuration for the profile σ'^{+}. These inequalities are sharp, hence they are insensitive to the effect of far away spins.

(3) *''Unfixing'' of the spins close to the origin.* Alter the previous set $O'^{+}_{U',M'}$ by allowing arbitrary values of $\sigma'_{\vec{x}'}$ for $\vec{x}'$ on a set Λ' formed by the origin and one of its neighbors. This corresponds to allowing fluctuations of these renormalized spins. This leads to a final set $O'^{+}_{\Lambda',U',M'}$ of profiles. It is clear, but boring to justify mathematically, that the probability distribution for the spins at Λ' will favor the configuration corresponding to the selected strip configuration. (See the discussion in [[8] Section 4.2].)

The upshot of this argument is therefore the following. Let $O'^{+}_{\Lambda',U',M'}$ and $\hat{O}'^{+}_{\Lambda',U',M'}$ be the families of profiles obtained by the above procedure for two different strip configurations (4.5). Then the preceding argument shows that there exists a $c>0$ such that

$$\sigma' \in O'^{+}_{\Lambda',U',M'}, \omega' \in \hat{O}'^{+}_{\Lambda',U',M'} \Rightarrow |\rho'_{\Lambda'}(\sigma') - \rho'_{\Lambda'}(\omega')| \geqslant c \tag{4.8}$$

for all large enough $U' \supset \Lambda'$ for $M' \supset U'$ sufficiently large. This proves that a violation of Eq. (3.10) is obtained when the profile inside U' is close to the ''all-0'' configuration. The fact that we have introduced another set M' is a concession to mathematical rigor: In this way the violation involves *open* sets of configurations and this ensures that the phenomenon is *essential* in probabilistic terms, that is, it cannot be avoided by redefining probability weights in sets of measure zero (open sets have nonzero renormalized measure because so do the initial measures, and smooth momentum transformation are continuous).

We see that the argument is insensitive to the presence of a magnetic field (because the constrained system is asked to have small magnetization), thus we are proving non-Gibbsianness for low temperatures but *arbitrary magnetic field*.

V. COMMENTS AND CONCLUSIONS

The present example of non-Gibbsianness as a consequence of momentum-space transformations confirms the suspicion of Griffiths [12] that ''no peculiarities of this sort have been found . . . , which may merely reflect the fact that no one has looked for them!'' Nevertheless, one should not draw too radical conclusions from this occurrence. On the practical side, the main implication of non-Gibbsianness is that one has to be very careful in designing renormalization group transformations. This is in complete agreement with what the founders and various practitioners of renormalization-group methods have been saying all along.

Indeed, already Wilson and Kogut in their classic review

emphasized ''Otherwise, [that is, nonperturbatively], the locality of [the renormalized interactions] is a nontrivial problem, which will not be discussed further'' [[14], p. 145]. And more explicitly, Fisher in his ''Renormalization Group Desiderata'' listed the conditions needed for a successful renormalization scheme in Hamiltonian space: ''A renormalization group for a space of Hamiltonians should satisfy the following: (A) Existence *in* the thermodynamic limit, . . . , (C) Spatial locality, . . . , one should be able to identify the same regions of space and associated local variables before and after the transformation'' [[1], Section 5.4.2].

Our example adds to the numerous instances showing that perversely or sloppily designed transformations can lead people into trouble. As Goldenfeld points out in his book *Lectures on Phase Transitions and the Renormalization Group* [[3], p. 268], ''It is dangerous to proceed without thinking about the physics.'' The moral is, then, that renormalization transformations must be carefully crafted and case-tailored. Already Wilson, as quoted in [[23], p. 492], warned ''One cannot write a renormalization cookbook.''

On the foundational side, examples like the present one confirm the view expressed by Benfatto and Gallavotti [2] in the opening sentence of their book *Renormalisation Group*, ''The notion of Renormalisation Group is not well-defined.'' It is clear that the mathematical formalization of the method requires much more than a naive approach in terms of Hamiltonians and flows of coupling constants. In fact, the example of this paper illustrates some features pointing into promising directions for a better mathematical understanding of the renormalization-group framework.

First, our problematic profiles were configurations with small magnetization. At low temperatures, this corresponds to a large fluctuation from the typical behavior, in which the magnetization in a region of width L is either positive or negative of order $O(L^d)$. Renormalized effective interactions are known not to be adequate to describe such large values of the fluctuation field [20]; geometric expansions are much more suitable. This suggests to combine renormalization-group ideas with this type of expansion—cluster or polymer expansion—to circumvent the ill-definedness of the renormalized Hamiltonian. These expansions have indeed been successfully applied in the rigorous control of renormalization-group transformations of unbounded-spin systems [[19,20], and references therein]. A related approach, for bounded-spin systems, resorted to the renormalization of Peierls-like contours [24].

This observation supports the idea that spin variables may be the ''wrong'' variables and that the appropriate variables in the presence of first-order transitions are nonlocal variables such as contours. It should be pointed out, however, that the use of contours requires the consideration of different (contour) ensembles for separate phases. This would go against the usual renormalization-group description based on flows of parameters in spaces where the various parts of the phase diagram can be connected, at least in a neighborhood of the critical point. In this regard, the approach based on low-temperature contour variables provides at best a partial answer to the problem of rigorously justifying renormalization-group calculations.

The second feature of our example is that the violation of continuity was detected for a renormalized configuration that is rather atypical (for instance, it will never be generated in any reasonable numerical simulation scheme). This seems to be a systematic feature of most examples, and prompted Dobrushin to propose the study of these measures with techniques borrowed from the treatment of other known systems where it is necessary to exclude sets of ''bad'' configurations, namely unbounded spin systems and systems exhibiting Griffiths singularities. This has given rise to a healthy body of work [25–34]. As an upshot, a more general theory involving a wider class of allowable Hamiltonians has been proposed. This theory leads to the notion of ''weak Gibbsianness'' which seems a promising framework for a unified treatment. See, for instance, [35] for a review of results in this and related directions.

We think our result illustrates and clarifies to some extent the reason why finding a good renormalization-group scheme is such a nontrivial task, not only for strictly local but also for only approximately local transformations. We produced an example in the low-temperature regime, but the fact that the mechanisms of non-Gibbsianness are so similar for real-space and momentum-space transformations leads us to the conjecture that, as in real space, also in momentum-space one cannot trust that in general the critical region is free of problems.

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